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# Deduction in Matching Logic

Master's Thesis

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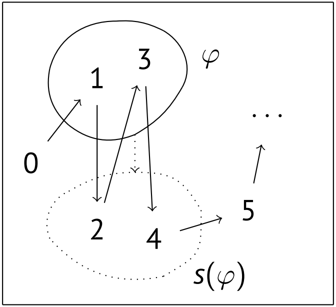
# Matching logic

## Intuition




$$\mathfrak{M} = (\mathbb{N}, \{\underline{0}, s^{\mathfrak{M}}\})$$

Figure: Matching the pattern $\varphi \equiv (\underline{1} \vee \underline{2} \vee \underline{3}) \wedge \neg x$ with $x := 2$.

# Matching logic
## Motivation

Matching logic (ML) is designed for reasoning about operational semantics.

1. Operational semantics can be defined as a ML theory $\Gamma$.
2. Then we can prove various program properties, e.g.,

$$\Gamma \vdash \langle \varphi_{state}, x := 1 \rangle \to \langle \varphi_{state}, y := 1 \rangle.$$

| | |
|---|---|
| (PT) | $\varphi$ if $\varphi$ is a propositional tautology over patterns |
| (MP) | $\dfrac{\varphi_1 \quad \varphi_1 \to \varphi_2}{\varphi_2}$ |
| $(\forall)$ | $(\forall x.\, \varphi_1 \to \varphi_2) \to (\varphi_1 \to \forall x.\, \varphi_2)$ if $x \notin FV(\varphi_1)$ |
| (Sub) | $(\forall x.\, \varphi) \to \varphi[y/x]$ |
| (Gen) | $\dfrac{\varphi}{\forall x.\, \varphi}$ |
| (Propagation$_\bot$) | $C_\sigma[\bot] \to \bot$ |
| (Propagation$_\vee$) | $C_\sigma[\varphi_1 \vee \varphi_2] \to (C_\sigma[\varphi_1] \vee C_\sigma[\varphi_2])$ |
| (Propagation$_\exists$) | $C_\sigma[\exists x.\varphi] \to \exists x.C_\sigma[\varphi]$ if $x \notin FV(C_\sigma[\exists x.\varphi])$ |
| (Framing) | $\dfrac{\varphi_1 \to \varphi_2}{C_\sigma[\varphi_1] \to C_\sigma[\varphi_2]}$ where $C_\sigma$ is a single symbol context. |
| (Ex) | $\exists x.\, x$ |
| (Singleton) | $\neg(C_1[x \wedge \varphi] \wedge C_2[x \wedge \neg\varphi])$ where $C_1, C_2$ are nested symbol contexts. |

Figure: System $\mathcal{H}$

# Properties of System $\mathcal{H}$

## Soundness

**Theorem (Chen and Roşu, 2019b)**

System $\mathcal{H}$ is sound, i.e., $\Gamma \vdash \varphi$ implies $\Gamma \models \varphi$.

Is System $\mathcal{H}$ complete?

# Properties of System $\mathcal{H}$
## Completeness

Let $\lceil \cdot \rceil$ be some unary symbol. We introduce the axiom

$$\text{(Definedness)} \qquad \forall x.\, \lceil x \rceil$$

**Theorem (Chen and Roşu, 2019b)**

Let $\Gamma$ be a theory containing (Definedness). If $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$.

# Problem

Is System $\mathcal{H}$ complete w.r.t. *all* theories?

1. Not all theories contain (Definedness).
2. We want to know why (Definedness) is so important.
3. ML has deep connections with other logics.

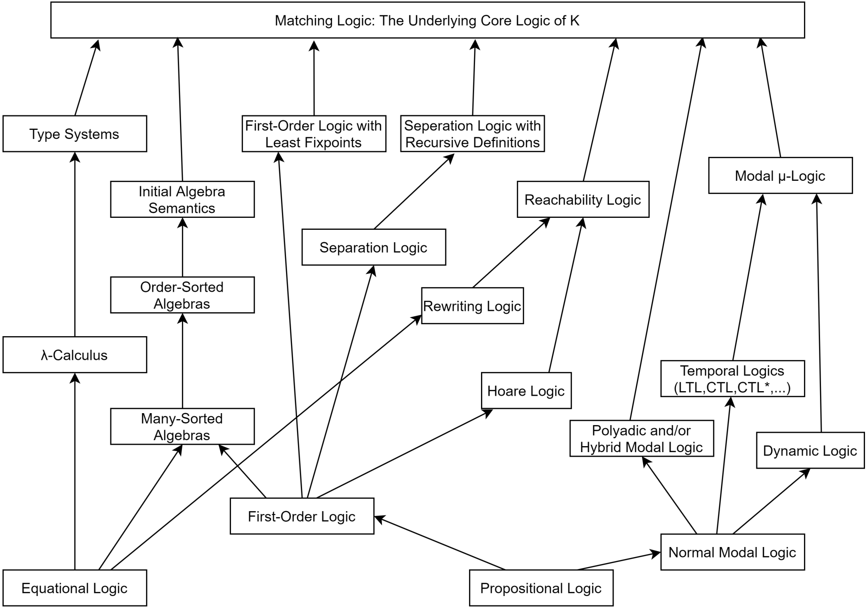



Figure: Logics successfully embedded into ML (Chen and Roşu, 2019a).

# New results

We identified a new characterization of completeness for System $\mathcal{H}$:

**Theorem**

Let $\lceil \cdot \rceil$ be some *fresh* symbol not contained in $\Gamma$ and $\varphi$. The following two statements are equivalent.

1. $\Gamma \models \varphi$ implies $\Gamma \vdash \varphi$.
2. $\Gamma \cup \{\forall x. \lceil x \rceil\} \vdash \varphi$ implies $\Gamma \vdash \varphi$.

# New results

Our characterization led to new results:

**Theorem**

System $\mathcal{H}$ is complete iff System $\mathcal{H}$ is complete w.r.t. *finite theories*.

**Theorem**

System $\mathcal{H}$ is complete w.r.t. the fragment of ML without symbols.

**Theorem (Compactness)**

Let $\Gamma$ be any theory. Then $\Gamma$ is satisfiable iff every finite subset of $\Gamma$ is satisfiable.

# New results

We showed new connections with FOL:

**Theorem (Full FOL embedding)**

Let $\Phi$ be a FOL theory. Then there exists a ML theory $\Gamma$ such that $\Phi \models_{\text{FOL}} \varphi$ iff $\Phi \cup \Gamma \models_{\text{ML}} \varphi$ for every FOL formula $\varphi$.

**Theorem (Henkin's characterization in ML)**

Under certain technical assumptions, the following two are equivalent.

1. System $\mathcal{H}$ is complete.
2. Every consistent theory has a model.

# New results

We devised a new[1] *canonical model* construction with the following property:

**Theorem (Canonical model)**

Let $\Gamma$ be any theory containing (Definedness). If $\Gamma \nvdash \varphi$, then we can construct a model $\mathfrak{M}$ such that $\mathfrak{M} \models \Gamma$ and $\mathfrak{M} \not\models \varphi$.

[1] Inspired by (Blackburn and Tzakova, 1998), (Chen and Roşu, 2019b).

# Future work

- Are there any other techniques for *conservative extensions*?
  Recall that for a fresh $\lceil \cdot \rceil$, it suffices to prove

$$\Gamma \cup \{\forall x.\, \lceil x \rceil\} \vdash \varphi \text{ implies } \Gamma \vdash \varphi.$$

- Does the connection with FOL still carry some answers?
- Are we able to extend our canonical models to all theories?

# Conclusion

We have

1. identified a new characterization of completeness,
2. proved that completeness can be reduced to *finite theories*,
3. proved the *compactness theorem*,
4. showed some new classes of theories for which $\mathcal{H}$ is complete,
5. discovered new connections with FOL,
6. devised a new technique for constructing canonical models.

# Bibliography


Blackburn, Patrick and Miroslava Tzakova (July 1998). “Hybrid completeness”. In: *Logic Journal of the IGPL* 6.4, pp. 625–650. issn: 1367-0751. doi: `10.1093/jigpal/6.4.625`. url: `https://academic.oup.com/jigpal/article-pdf/6/4/625/1878635/060625.pdf`.

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